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On Radius Critical Graphs

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Abstracts

The radius r(G) of a connected graph G is defined by:

$$r(G) = \min_{u \in V(G)} \max_{v \in V(G)-u} d(u,v)$$

where d(u,v) is the length of the shortest path in G between u and v. G is radius-critical if deleting any vertex from G reduces its radius by 1. In this paper we relate this notion to the concepts of eccentricity and give characterizations of edge-maximal 3-radius-critical graphs. In particular, we show that every edge-maximal 3-radius-critical graph is edge-maximum.

1. Introduction

In this section we introduce some notions and obtain some preliminary results on radius critical graphs. Let G be a connected graph and let G' be the graph obtained by deleting some given vertex v of G. Let d and d' be the corresponding distance functions. If u and w are vertices in G then d(u,w) is the length of the shortest path from u to w in G. Since G'

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need not be connected, the situation $d'(u,w) = +\infty$ is possible.

It is easily verified that:

$$d'(u,w) \geqq d(u,w) \quad \text{for all } u,w \in V(G')$$

The <u>eccentricity</u> $e(u)$ of a vertex u in a graph G is defined by

$$e(u) = \max \{d(u,w) : w \in G\}.$$

Let $e'(u)$ be the eccentricity of vertex u in G'. We denote by $N_G(u)$ the <u>open</u> (nearest) <u>neighborhood</u> of a vertex u, that is the set of vertices adjacent to u. $N_G[u]$ denotes the <u>closed</u> (nearest) <u>neighborhood</u> which is defined by

$$N_G[u] = N_G(u) \cup \{u\}.$$

The <u>furthest neighborhood</u> $FN_G(u)$ of a vertex u is defined by

$$FN_G(u) = \{w \in V(G) \mid d(u,w) = e(u)\}.$$

A vertex v in $FN_G(u)$ is called a <u>furthest neighbor</u> of u. In case v is the unique furthest neighbor of u, we have:

$$e'(u) = e(u) - 1.$$

The <u>furthest neighbor graph</u> $FN(G)$ of a graph G is defined on $V(G)$ where uv is an edge of $FN(G)$ if and only if $u \in FN(v)$ or $v \in FN(u)$. The radius $rad(G)$ of a graph G is defined by

$$rad(G) = \min\{e(u) \mid u \in V(G)\}.$$

For any connected graph G and non cut vertex v we have

(1) $rad(G') \geqq rad(G) - 1.$

There is, however, no reasonable upper bound on the radius of G'. If G is the join of the path of size $2n$ and an extra vertex, that is, $p_{2n} + \{v\}$, then

$$rad(G) = 1 \quad \text{and} \quad rad(G') = n.$$

The inequality (1) leads to the definition of the following class of graphs.

<u>Definition</u>. A block G is <u>radius-critical</u> if for each $v \in V(G)$:

$$rad(G') = rad(G) - 1.$$

An <u>r-radius-critical</u> graph is a radius-critical graph with radius r. The above discussion leads to the following characterization of radius-critical graphs.

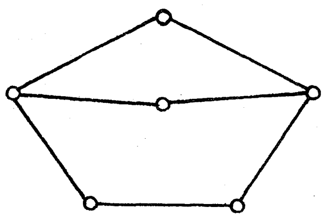
<u>Lemma 1</u>: G is radius-critical if and only if

(i) each vertex in G has a unique furthest neighbor

and

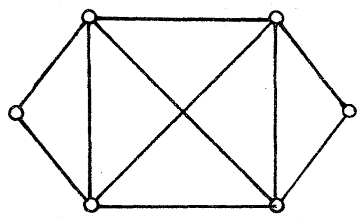
(ii) each vertex in G has the same eccentricity, (equi-eccentric).

The graphs (i) and (ii) in figure 1 show the necessity of the two conditions:



equi-eccentric but non-unique furthest neighbors.

(i)



unique furthest neighbor but not equi-eccentric.

(ii)

Fig. 1

In fact, condition (i) is not strong enough to make G a block, as paths of even order satisfy (i) but not (ii). Further properties of equi-eccentric graphs may be found in Ando et al [1].

2. Polarities and Radius-critical graphs.

We begin with the following definition:

Definition. ψ is a *polarity*[1] on a connected graph G if ψ is a fixed point free involution on V(G) such that

$$d(u,v) = d(u,\psi(u)) \Rightarrow v = \psi(u) \quad \text{for all } u \in V(G).$$

Let G be an equi-eccentric block such that each vertex has a unique furthest neighbor. Then it is easily verified that

(2) $\psi(u) = FN_G(u)$ for all $u \in V(G)$

defines a polarity. The next result shows that these are in fact the only polarities on blocks.

Theorem 1. There is a polarity on a block G if and only if G is radius-critical.

Proof: ($\Leftarrow$) By lemma 1, G is radius critical if and only if it is equi-eccentric and each vertex has a unique furthest neighbor. As remarked above, (2) defines a polarity.

($\Rightarrow$) Let G be a block with polarity ψ. First suppose for some vertex u, $e(u) = 1$. Then

$$d(u,\psi(u)) = 1.$$

and G must be K_2, since

$$d(u,v) = 1 \quad \Rightarrow \quad v = \psi(u).$$

Thus we may assume that $e(u) \geqq 2$ for all vertices u. Since G is a block, for any k smaller than e(u), there must be at least two vertices v,w such that

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1. The authors are grateful to H. Enomoto for suggesting this term.

$$d(u,v) = d(u,w) = k.$$

Thus, by the definition of polarity

$$d(u,\psi(u)) = e(u) \quad \text{and} \quad FN_G(u) = \{\psi(u)\}.$$

Consider some vertex w in $N_G(\psi(u))$. Clearly

$$e(w) \geqq e(u) - 1.$$

The inequality is in fact strict, since otherwise $\psi(w) = u$, contradicting the assumption that ψ is an involution. Therefore,

$$e(w) \geqq e(u) \quad \text{for all } w \in N_G(\psi(u)).$$

This implies that G is equi-eccentric. By the lemma, G must be radius-critical.

Corollary. G is radius-critical if and only if

$$FN(G) = nK_2, \quad n \geqq 2.$$

<u>Proof</u>: It is easily verified that if $FN(G) = nK_2$ and $n \geqq 2$, then G must be a block. The statement then follows from theorem 1.

3. <u>3-radius-critical graphs</u>

In this section we study edge-maximal 3-radius-critical graphs. We show that a 3-radius-critical graph is edge-maximal if and only if it is edge-maximum. Finally we obtain a chacterization of 3-radius-critical graphs.

We first obtain a bound on the maximum degree $\Delta(G)$, of a 3-radius-critical graph G of even order p. Since every vertex of G has eccentricity 3,

$$\psi(N[v]) \cap N[v] = \phi \quad \text{for all } v \in V(G).$$

Otherwise let x be a vertex not only in N[v] but also in ψ (N[v]).

Then $d(v,\psi(v)) \leqq 2$, a contradiction. This implies that

$$\deg(v) \leqq (p-2)/2.$$

Define $Hp = K_{p/2,p/2} - (p/2)K_2$. That is, Hp is the complete (p/2, p/2) bipartite graph minus a one-factor. It is easily verified that Hp is 3-radius-critical, and, by the above remark, also edge-maximum. Thus 3-radius-critical edge-maximum graphs are (p-2)/2-regular.

Lemma 2. Let G be a 3-radius-critical graph. Suppose u and v are non-adjacent vertices satisfying:

(i) $u \in V(G) - \{N[v] \cup [\psi(v)]\}$

(ii) $\psi(u) \notin N[v]$

then joining u and v by an edge leaves a 3-radius-critical graph.

Proof: Let H be the graph formed from G by joining u and v, then

$$d_H(x,y) \leqq d_G(x,y) \quad \text{for all } x,y \in V(G).$$

So we need only show that

(3) $d_H(x,y) \leqq 2$ implies that $d_G(x,y) \leqq 2$.

Case 1. $d_H(x,y) = 1$.

Either xy is an edge in G, in which case(3) is trivial, or xy is the edge uv. But if xy is the edge uv, then $d_G(x,y) \leqq 2$ because of condition (ii).

Case 2. $d_H(x,y) = 2$

Let w be adjacent to x and y in H. If w is adjacent to x and y in G, (3) is immediate. Thus we may assume that xw is the new edge. Suppose x = u, w = v. Then condition (ii) implies that $y \neq \psi(u)$. Thus

$$d_G(x,y) = d_G(u,y) = 2.$$

Otherwise $x = v$, $w = u$. But then the condition (i) that $u \in N[\psi(v)]$ implies $y \nsim \psi(v)$. Thus

$$d_G(x,y) = d_G(v,y) = 2.$$

Theorem 2. A 3-radius-critical graph G is edge-maximal if and only if it is edge-maximum.

Proof: Suppose G is an edge-maximal 3-radius-critical graph and that x is a vertex of degree less than $(p-2)/2$. Then we will find two vertices satisfying the conditions of lemma 2, yielding a contradiction to the edge maximality of G. Let

$$W = V(G) - \{N_G[x] \cup N_G[\psi(x)]\}.$$

The degree condition on x implies that W is not empty. Suppose $\psi(W) \not\subseteq N[x]$, and choose $y \in W$ such that $\psi(y) \notin N[x]$. Then setting $u = y$ and $v = x$, the conditions of lemma 2 are satisfied.

Otherwise, $\psi(W) \subseteq N[x]$ so $\psi(W) \not\subseteq N[\psi(x)]$. In this case, choose $y \in W$ such that $\psi(y) \in N[\psi(x)]$. Then setting $u = y$ and $v = \psi(x)$, the conditions of lemma 2 are satisfied. Thus the theorem follows.

We remark that edge-minimal 3-radius-critical graphs are not necessarily edge-minimum. Consider, for example, the edge-minimal 3-radius-critical graph $H_5 = K_{5,5} - 5K_2$, which is in fact edge-maximum!

Before giving a final result on edge-maximum 3-radius-critical graphs, we need a new definition.

<u>Definition</u>. The distance two graph G_2 of a graph G is defined on V(G), where uv is an edge of G_2 if and only if $d_G(u,v) = 2$.

<u>Theorem 3</u>. G is an edge-maximal 3-radius-critical graph of order p if and only if G is Hp or G_2 is an edge-maximal 3-radius-critical graph.

<u>Proof</u>: If G is Hp the theorem is immediate. So let G be any edge-maximal 3-radius-critical graph that is not Hp. Let ψ be the polarity of G and set

$$E' = \{uv \in E(\bar{G}) \mid \psi(u) = v\}.$$

Then it may be verfied that $G_2 = \bar{G} - E'$. We will verify that G_2 is also 3-radius-critical edge-maximal.

<u>Case 1</u>. $d_G(u,v) = 3$.

In this case $v = \psi(u)$, $uv \in E'$ and thus $d_{G_2}(u,v) \geqq 2$. Further, G is also edge maximum, so

$$V(G) = N_G[u] \cup N_G[\psi(u)]$$

since G is (p-2)/2-regular. This implies that

$$N_G[u] \cap N_G[\psi(u)] = \phi,$$

and hence $d_{G_2}(u,v) \geqq 3$. On the other hand, G_2 is also (p-2)/2-regular and is not $K_{p/2} \cup K_{p/2}$, since $G \neq Hp$. Thus G_2 is connected and $d_{G_2}(u,v) = 3$.

<u>Case 2</u>. $d_G(u,v) = 2$.

By definition, $d_{G_2}(u,v) = 1$.

<u>Case 3</u>. $d_G(u,v) = 1$.

Since ψ is a polarity,

$$d_G(u,\psi(v)) = d_G(v,\psi(u)) = 2.$$

Hence $u\psi(v)$ and $v\psi(u)$ are edges in G_2 and $d_{G_2}(u,v) = 2$.

We now see that G_2 had radius 3 and every vertex has degree $(p-2)/2$. Further ψ is a polarity on G_2 and G_2 is therefore edge-maximal radius-critical. Under these conditions, we may interchange the roles of G and G_2 in the above case analysis to see that $(G_2)_2 = G$, or in other words, the distance two graph of G_2 is G. This proves the sufficiency of the statement of theorem 3.

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